TRINITY COLLEGE

IN NOMINE DOMINI

Semester One Examination, 2020

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(5 marks)**

The function $f(z)$ is of degree 4 and has factors $z-4-i$ and $z+3i$.

(a) Determine $f(z)$ in the form $z^4+az^3+bz^2+cz+d$, where $\{a,b,c,d\} \in \mathbb{R}$. (3 marks)

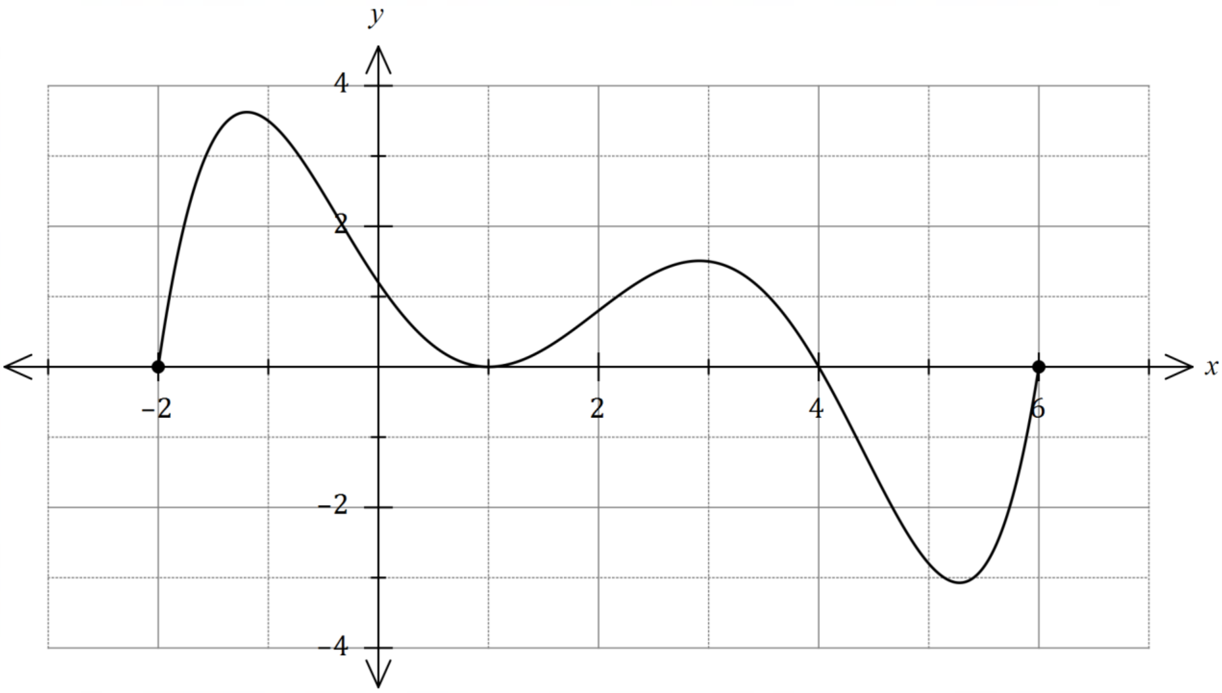
Solution
Two other factors must be $z-4+i$ and $z-3i$. $(z-4-i)(z-4+i)=z^2-8z+17$ $(z-3i)(z+3i)=z^2+9$ $(z^2+9)(z^2-8z+17)=z^4-8z^3+26z^2-72z+153$
Specific behaviours
✓ uses conjugate roots to obtain all factors ✓ indicates product of all factors ✓ correct $f(z)$

(b) Explain whether your answer to part (a) would change if the coefficients of the polynomial $f(z)$ were not restricted to real numbers. (2 marks)

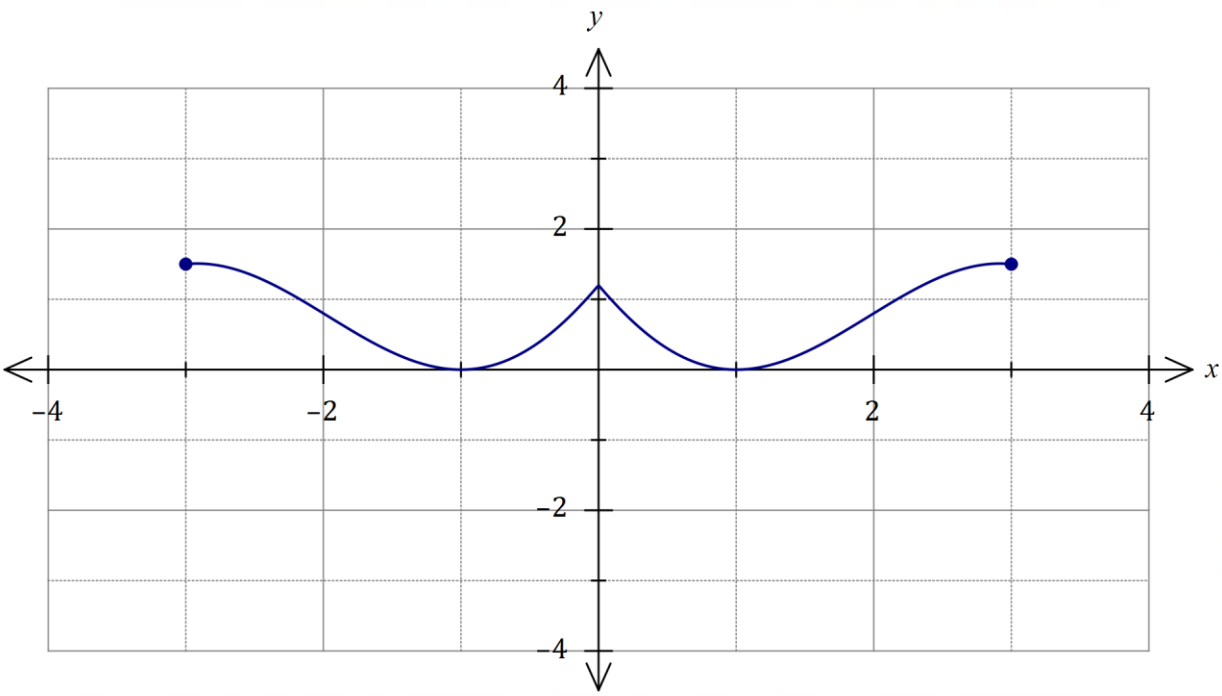
Solution
When $\{a,b,c,d\} \in \mathbb{R}$ then there is a unique solution as the roots will be in conjugate pairs. Without this restriction there is an infinite number of choices for the other two factors and so answer would very likely be different.
Specific behaviours
✓ indicates unique solution for real coefficients ✓ indicates large number of possibilities otherwise

Question 10 **(8 marks)**

The graph of $y = f(x)$ is shown below over the domain $-2 \leq x \leq 6$.

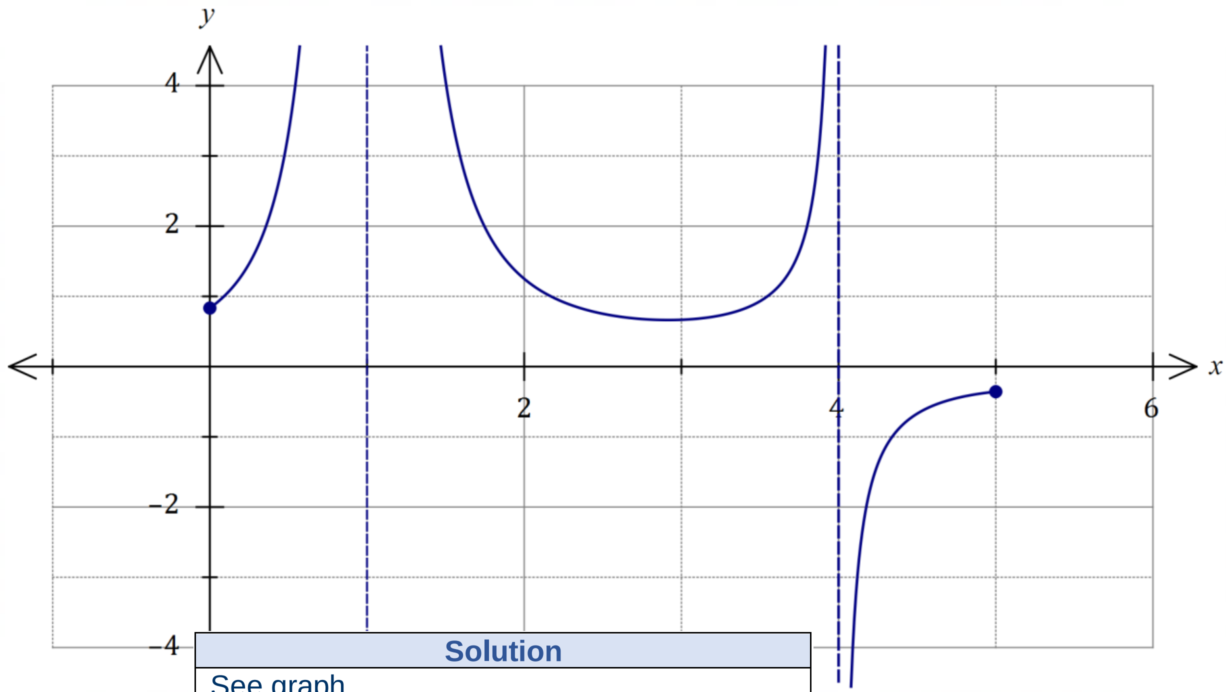


(a) Sketch the graph of $y = f(|x|)$ over the domain $-3 \leq x \leq 3$ on the axes below. (2 marks)



Solution
See graph
Specific behaviours
✓ cusp and curvature $-1 < x < 1$
✓ endpoints and symmetrical curve

(b) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below over the domain $0 \le x \le 5$. (4 marks)



Solution
See graph
Specific behaviours
✓ endpoint and curvature $0 \le x < 1$
✓ endpoint and curvature $4 < x \le 5$
✓ indicates vertical asymptotes $x = 1, x = 4$
✓ minimum and curvature $1 < x < 4$

(c) List the equations of all asymptotes of the graph of $y = \dfrac{1}{f(|x|)}$ when drawn over the domain $-6 \le x \le 6$. (2 marks)

Solution
Zeroes of $f(x)$ for $0 \le x \le 6$ at $x = 1, 4, 6$ Hence six asymptotes: $x = \pm 1, \quad x = \pm 4, \quad x = \pm 6$
Specific behaviours
✓ four or more correct asymptotes
✓ lists exactly six asymptotes, all correct

Question 11 **(8 marks)**

Drone A and drone B move with constant velocities and relative to the origin O have initial positions $(-4, 22, 2)$ and $(5, 15, 3)$ respectively, where distances are in metres.

One second later, the position of A is $(-1, 20, 3)$ and the position of B is $(1, 14, 8)$.

(a) Determine a position vector relative to the origin for each drone after t seconds. (3 marks)

Solution
$\mathbf{v}_A = \begin{pmatrix} -1 \\ 20 \\ 3 \end{pmatrix} - \begin{pmatrix} -4 \\ 22 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_B = \begin{pmatrix} 1 \\ 14 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ 15 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$ $\mathbf{r}_A = \begin{pmatrix} -4 \\ 22 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{r}_B = \begin{pmatrix} 5 \\ 15 \\ 3 \end{pmatrix} + t\begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$
Specific behaviours
✓ derives velocity vectors   ✓ position vector for A ✓ position vector for B

(b) Determine an expression for the distance between the two drones at any time t, $t \geq 0$. (3 marks)

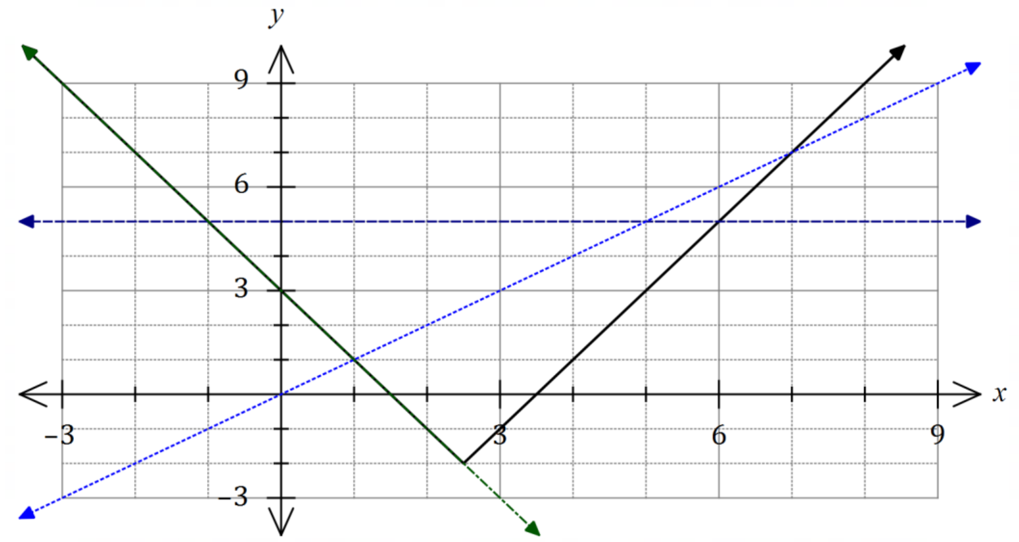
Solution
$\mathbf{r}_B - \mathbf{r}_A = \begin{pmatrix} 9 - 7t \\ -7 + t \\ 1 + 4t \end{pmatrix}$ $s = \sqrt{(9 - 7t)^2 + (t - 7)^2 + (1 + 4t)^2}$ $= \sqrt{66t^2 - 132t + 131}$
Specific behaviours
✓ difference of position vectors   ✓ indicates magnitude of vector   ✓ simplified expression

(c) Determine the minimum distance between the drones. (2 marks)

Solution
$\dfrac{ds}{dt} = \dfrac{66(t-1)}{\sqrt{66t^2 - 132t + 131}}$ $\dot{s} = 0 \Rightarrow t = 1, \quad s(1) = \sqrt{65} \approx 8.06 \text{ m}$
Specific behaviours
✓ time when minimum   ✓ correct distance

Question 12 **(7 marks)**

The graph of $f(x) = |ax + b| + c$ is shown below.



(a) Determine all possible values of the constants a, b and c. (3 marks)

Solution
$c = -2$ Either $\{a = 2, b = -5\}$ or $\{a = -2, b = 5\}$
Specific behaviours
✓ value of c ✓ one correct set for a, b ✓ both correct sets for a, b

(b) Using the graph, or otherwise, solve

(i) $f(x) = 5$. (1 mark)

Solution
$x = -1, \quad x = 6$
Specific behaviours
✓ correct values

(ii) $f(x) = x$. (1 mark)

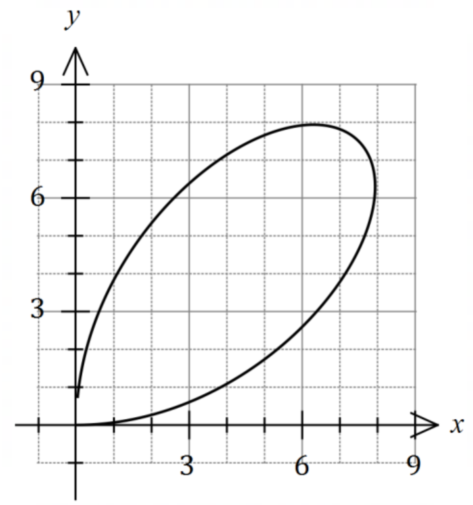
Solution
$x = 1, \quad x = 7$
Specific behaviours
✓ correct values

(iii) $f(x) + 2x = 3$. (2 marks)

Solution
$f(x) = 3 - 2x$ $x \leq 2.5$
Specific behaviours
✓ indicates sketch of line   ✓ correct inequality

Question 13 **(9 marks)**

The path of a particle with position vector $\mathbf{r}(t) = \dfrac{15t}{1+t^3}\mathbf{i} + \dfrac{15t^2}{1+t^3}\mathbf{j}$ metres is shown below, where t is the time in seconds and $t \geq 0$.



(a) Determine the initial velocity of the particle. (2 marks)

Solution
$\mathbf{v}(t) = \dfrac{15 - 30t^3}{(1+t^3)^2}\mathbf{i} + \dfrac{30t - 15t^4}{(1+t^3)^2}\mathbf{j}$ $\mathbf{v}(0) = 15\mathbf{i}$ m/s
Specific behaviours
✓ expression for velocity   ✓ initial velocity

(b) Determine the velocity of the particle at the instant, $t > 0$, when it is moving parallel to the x-axis. (2 marks)

Solution
Require **j**-coefficient of velocity to be zero: $30t - 15t^4 = 0 \Rightarrow t = \sqrt[3]{2}$ $\mathbf{v}(\sqrt[3]{2}) = -5\mathbf{i}$ m/s
Specific behaviours
✓ solves for t ✓ velocity

(c) Explain whether the particle will return to its initial position. (2 marks)

Solution
The initial position is at the origin, so the answer is no. The particle will get very close as $t \to \infty$ but neither coefficient of the position vector will ever reach zero, apart from initially.
Specific behaviours
✓ indicates initial position ✓ explains will get close, but never returns

(d) Observing that $y - xt = 0$, show that the Cartesian equation for the path of the particle can be expressed in the form $x^3 + y^3 = kxy$ and state the value of the constant k.
(3 marks)

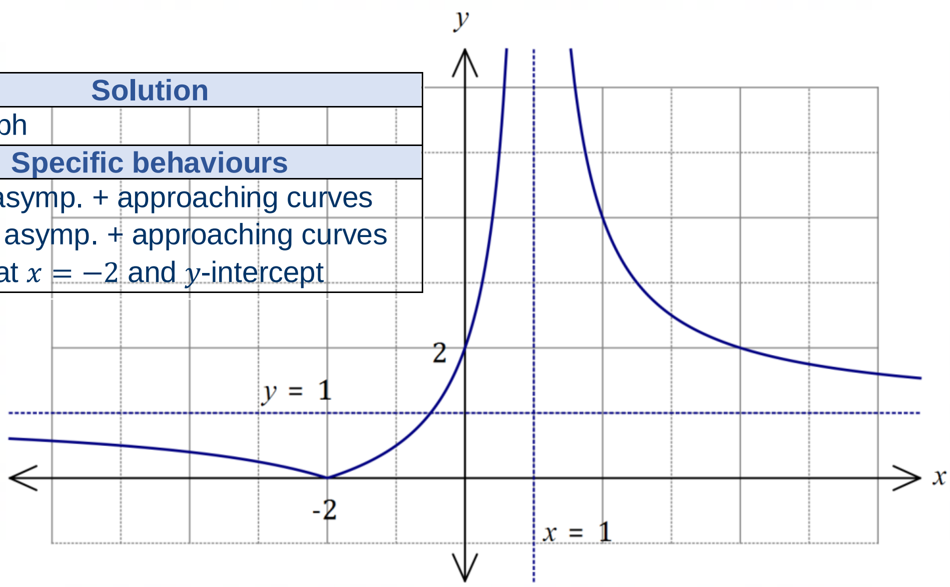
Solution
$t = \frac{y}{x}$ $x = \dfrac{15\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^3}$ $x\left(\dfrac{x^3+y^3}{x^3}\right) = \dfrac{15y}{x}$ $x^3 + y^3 = \dfrac{15yx^3}{x^2}$ $x^3 + y^3 = 15xy \Rightarrow k = 15$
Specific behaviours
✓ expresses x or y term using $\frac{y}{x}$ ✓ correctly obtains expression containing $x^3 + y^3$ ✓ manipulates into required form and states value of k

Question 14 **(7 marks)**

Let $f(x) = \left|\dfrac{x+2}{x-1}\right|$.

(a) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)

Solution
See graph
Specific behaviours
✓ vert. asymp. + approaching curves ✓ horiz. asymp. + approaching curves ✓ cusp at $x = -2$ and y-intercept



(b) State the range of $f(x)$. (1 mark)

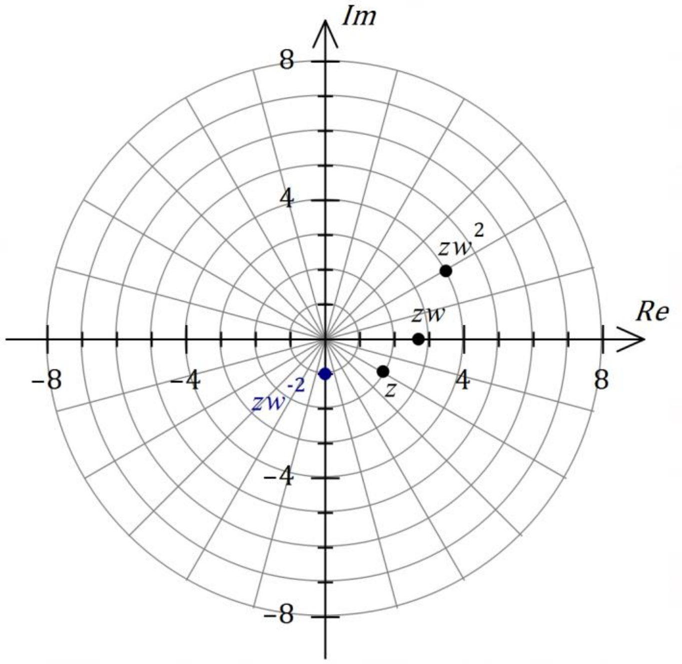
Solution
$R_f = \{y \in \mathbb{R}, y \geq 0\}$
Specific behaviours
✓ states $y \geq 0$

(c) The domain of f is restricted to $-2 \leq x < b$ so that f^{-1} is a function. State the value of the constant b so that the domain of f is as large as possible and determine the domain and range for f^{-1}. (3 marks)

Solution
$b = 1$ $D_{f^{-1}} = R_f = \{x \in \mathbb{R}, x \geq 0\}$ $R_{f^{-1}} = D_f = \{y \in \mathbb{R}, -2 \leq y < 1\}$
Specific behaviours
✓ value of b ✓ domain ✓ range

Question 15 **(8 marks)**

The complex numbers z, zw and zw^2 are represented on the Argand diagram below.



(a) Express z exactly in the form $a + bi$. (2 marks)

Solution
$z = 2\,\text{cis}\left(-\frac{\pi}{6}\right) = \sqrt{3} - i$
Specific behaviours
✓ polar form ✓ Cartesian form

(b) Determine the modulus and argument of zw^5. (4 marks)

Solution
$2 \times \lvert w \rvert^2 = 4 \Rightarrow \lvert w \rvert = \sqrt{2}$ $\arg(w) = \frac{\pi}{6}$ $zw^5 = 2\,\text{cis}\left(-\frac{\pi}{6}\right) \times \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{6}\right)\right)^5$ $= 8\sqrt{2}\,\text{cis}\left(\frac{2\pi}{3}\right)$ Modulus: $8\sqrt{2}$, Argument: $\frac{2\pi}{3}$
Specific behaviours
✓ modulus of w ✓ argument of w (accept $\pm 2n\pi$) ✓ forms product ✓ states modulus and argument

(c) Determine zw^{-2} and plot and label this point on the Argand diagram. (2 marks)

Solution
$zw^{-2} = \text{cis}\left(-\frac{\pi}{2}\right) = -i$
Specific behaviours
✓ correct value in any form ✓ correctly plots point

Question 16 **(9 marks)**

(a) Let $f(x) = \dfrac{x^2 - 4x - 2}{x - 1}$.

(i) Briefly describe the feature of the rule for $f(x)$ that indicates the graph of $y = f(x)$ will have an oblique (slanted) asymptote. (1 mark)

Solution
The degree of the polynomial in the numerator is one higher than that of the polynomial in the denominator.
Specific behaviours
✓ reasonable explanation

(ii) Determine the equations of all asymptotes of the graph of $y = f(x)$. (3 marks)

Solution
Vertical: $x = 1$ Oblique: $f(x) = \dfrac{x^2 - x}{x - 1} + \dfrac{-3x + 3}{x - 1} + \dfrac{-5}{x - 1}$ $= x - 3 - \dfrac{5}{x - 1}$ Hence asymptotes are $x = 1$ and $y = x - 3$.
Specific behaviours
✓ vertical asymptote ✓ expresses $f(x)$ to expose oblique asymptote ✓ oblique asymptote

(b) Let $g(x) = \dfrac{(x-2)(x+3)}{x^2+1}$.

(i) State the equation of the horizontal asymptote of the graph of $y = g(x)$. (1 mark)

Solution
$y = 1$
Specific behaviours
✓ asymptote

(ii) State the values of $g(6), g(7)$ and $g(8)$. (1 mark)

Solution
$g(6) = \dfrac{36}{37} \approx 0.97, \quad g(7) = 1, \quad g(8) = \dfrac{66}{65} \approx 1.02$
Specific behaviours
✓ correct values

(iii) Use your previous two answers to explain why the graph of $y = g(x)$ must have a local maximum to the right of $x = 7$. (3 marks)

Solution
As $g(x)$ increases through $x = 7$, y is increasing and the curve cuts the horizontal asymptote $y = 1$. However, as $x \to \infty$, $y \to 1$ and since g is continuous for all x (has no vertical asymptotes) then at some point where $x > 7$ the curve must start to decrease to return to the asymptote and so a local maximum must exist. *NB Students may also use a sketch as part of their response, so long as it specifically uses the results from (i) and (ii).*
Specific behaviours
✓ indicates g increases through asymptote ✓ states g is continuous throughout ✓ explains why g must then decrease

Question 17 **(8 marks)**

Plane Π has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(a) Show how to deduce that the Cartesian equation of plane Π is $x - 5y - 3z = 1$. (3 marks)

Solution

Require $\mathbf{r} \cdot \mathbf{n} = k$. Direction vectors lie in plane, so normal will be:

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$$

And $k = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} = 1$

Hence equation of plane is $x - 5y - 3z = 1$

Specific behaviours

✓ indicates two direction vectors lie in plane
✓ cross product to obtain normal
✓ dot product with point, derives equation

The line through $A(1, 4, 5)$ and point B is perpendicular to Π, and the midpoint of AB lies in Π.

(b) Determine the coordinates of B. (5 marks)

Solution

Equation of line through A, B is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$$

Intersection of line and plane when

$$\begin{pmatrix} 1 + t \\ 4 - 5t \\ 5 - 3t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} = 1$$

$$1 + t - 20 + 25t - 15 + 9t = 1 \Rightarrow t = 1$$

Since $t = 1$ for midpoint, then $t = 2$ for B:

$$\mathbf{r}_B = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$$

$$B(3, -6, -1)$$

Specific behaviours

✓ equation of line
✓ equation for intersection
✓ solves for t
✓ value of t for B
✓ coordinates of B

Question 18 **(8 marks)**

(a) Determine, in the form $r\operatorname{cis}\theta$, the solution of the equation $z^4 + 625i = 0$ that lies in the third quadrant of the complex plane ($-\pi < \theta < -\frac{\pi}{2}$). (4 marks)

Solution
$z^4 = -625i = 625\operatorname{cis}\left(-\frac{\pi}{2}\right)$ $z = 5\operatorname{cis}\left(-\frac{\pi + 4n\pi}{2\times 4}\right), n \in \mathbb{Z}$ $n = -1 \Rightarrow z = 5\operatorname{cis}\left(-\frac{5\pi}{8}\right)$
Specific behaviours
✓ equation in polar form   ✓ expression for roots   ✓ indicates correct choice for n ✓ correct solution

(b) Writing $5 - 12i = (a + bi)^2,\ \{a, b\} \in \mathbb{R}$, or otherwise, use an algebraic method that does not involve CAS to determine the square roots of $5 - 12i$. (4 marks)

Solution
$(a + bi)^2 = a^2 - b^2 + 2abi$ Real parts: $a^2 - b^2 = 5$ ... (1)   Imaginary parts: $2ab = -12$ ... (2)   Also, $\lvert a + bi\rvert^2 = \lvert 5 - 12i\rvert \Rightarrow a^2 + b^2 = 13$ ... (3)   From (1) and (3): $2a^2 = 18 \Rightarrow a = \pm 3$ From (2): $b = -12 \div 2(\pm 3) = \mp 2$ Hence square roots are $3 - 2i$ and $-3 + 2i$.
Specific behaviours
✓ equates real and imaginary parts   ✓ equates moduli   ✓ solves for one coefficient   ✓ correct square roots

Question 19 **(9 marks)**

The position vectors of particles A and B (in centimetres) at time t seconds, $t \geq 0$, are

$$\mathbf{r}_A = 7\mathbf{i} + 15\mathbf{j} + t(0.5\mathbf{i} - 2\mathbf{j}) \text{ and } \mathbf{r}_B = 5\mathbf{i} + 2\mathbf{j} + t\big((t-6)\mathbf{i} - \mathbf{j}\big).$$

(a) Show that A is moving with constant speed and determine this speed. (2 marks)

Solution
$\mathbf{v}_A = \begin{pmatrix} 0.5 \\ -2 \end{pmatrix}$, which is independent of t and hence constant. $\lvert\mathbf{v}_A\rvert = \dfrac{\sqrt{17}}{2} \approx 2.06\ \mathrm{cm/s}$
Specific behaviours
✓ obtains constant velocity vector   ✓ correct speed

(b) Determine the Cartesian path of B. (3 marks)

Solution
$y = 2 - t \Rightarrow t = 2 - y$ $x = 5 + t^2 - 6t$ $x = 5 + (2-y)^2 - 6(2-y)$ $x = y^2 + 2y - 3, \quad y \leq 2 \ \ (\text{as } y = 2 - t, t \geq 0)$
Specific behaviours
✓ expressions for x and y in terms of t ✓ eliminates t ✓ simplifies, noting domain

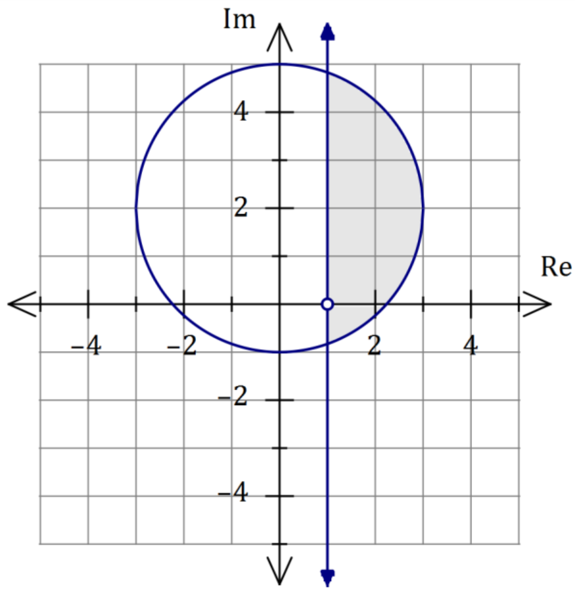
(c) Determine the position vector of the point where the paths of the particles cross. (4 marks)

Solution
Position of A after s seconds: $\mathbf{r}_A = \begin{pmatrix} 7 + 0.5s \\ 15 - 2s \end{pmatrix}$ Position of B after t seconds: $\mathbf{r}_B = \begin{pmatrix} 5 + t^2 - 6t \\ 2 - t \end{pmatrix}$ Hence require: $7 + 0.5s = 5 + t^2 - 6t$ $15 - 2s = 2 - t$ Solving simultaneously $(s, t > 0)$: $s = 10, t = 7$ Paths intersect at $\begin{pmatrix} 7 + 0.5(10) \\ 15 - 2(10) \end{pmatrix} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$
Specific behaviours
✓ positions using different variables for time   ✓ equates coefficients   ✓ solves for times   ✓ determines position vector for intersection

Alternative Solution
Cartesian path of A: $x = 7 + 0.5t \Rightarrow t = 2x - 14$ $y = 15 - 2t = 15 - 2(2x - 14)$ Solving simultaneously with eqn from (b): $\{x = 12, y = -5\}, \left\{x = \dfrac{161}{16}, y = \dfrac{11}{4}\right\}$ Ignore second solution since $y = \frac{11}{4} > 2$ using domain restriction from (b).   Hence paths intersect at $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$.
Specific behaviours
✓ Cartesian path for A ✓ solves simultaneously   ✓ checks for $y \leq 2$ ✓ states position vector of intersection

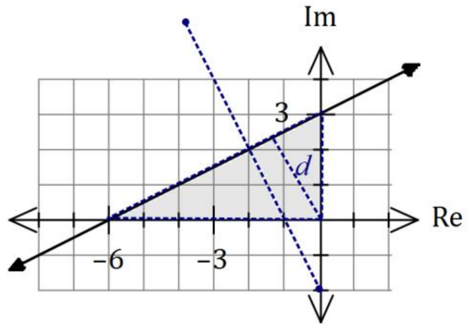
Question 20 **(8 marks)**

(a) Shade the region in the complex plane below that simultaneously satisfies $|z-2i| \leq 3$ and $-\frac{\pi}{2} \leq \arg(z-1) \leq \frac{\pi}{2}$. (4 marks)



Solution
See diagram
Specific behaviours
✓ circle $r=3$
✓ centre of circle
✓ line $x=1$
✓ identifies region

(b) The locus of $|z+2i| = |z+a+bi|$ in the complex plane is the straight line shown below, $\{a,b\} \in \mathbb{R}$.



(i) State the value of constant a and the value of constant b. (2 marks)

Solution
$a=4, \qquad b=-6$
Specific behaviours
✓ value of a
✓ value of b

(ii) Determine the minimum value of $|z|$ in exact form. (2 marks)

Solution
Hypotenuse of triangle: $\sqrt{6^2+3^2} = 3\sqrt{5}$ Area of triangle: $A = \frac{1}{2}(6)(3) = \frac{1}{2}(3\sqrt{5})(d)$ Minimum $|z| = d = \frac{6\sqrt{5}}{5}$
Specific behaviours
✓ indicates length representing minimum $|z|$
✓ exact value

Question 21 **(4 marks)**

Sphere S of radius 3 has its centre at the origin.

Line L has equation $\mathbf{r} = \begin{pmatrix} -k \\ k \\ -k \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, where k is a positive constant.

Prove that for L to be a tangent to S, then $k = \dfrac{3\sqrt{2}}{2}$.

Solution
$\lvert \mathbf{r} - \mathbf{0} \rvert = 3 \Rightarrow \left\lvert \begin{pmatrix} -k \\ k \\ -k \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right\rvert = 3$ $(\lambda - k)^2 + (2\lambda + k)^2 + (-2\lambda - k)^2 = 9$ $3\lambda^2 + 2k\lambda + k^2 - 3 = 0$ For tangent, require single solution for λ and so discriminant of quadratic in λ must be zero: $(2k)^2 - 4(3)(k^2 - 3) = 0$ $4k^2 - 12k^2 + 36 = 0$ $k^2 = \dfrac{9}{2} \Rightarrow k = \dfrac{3\sqrt{2}}{2}$
Specific behaviours
✓ substitutes equation of line into equation for sphere   ✓ equation for magnitude, simplified   ✓ explains requirement for one solution to quadratic   ✓ solves discriminant equation for positive k

End of questions

Supplementary page

Question number: _________